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An Analysis of Bonnor's Dipole Solution

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Abstract

Some years ago Bonnor presented a solution of the Einstein-Maxwell equations describing the field of a massive source carrying a magnetic dipole. In this paper I present an analysis of this solution. Bonnor's solution is interesting because of its close connection to the Kerr solution and also because it is not a member of the Weyl electromagnetic class.

Introduction

It has been long known that there is a close formal similarity between stationary exterior solutions of the Einstein equations and static magnetic solutions of the Einstein-Maxwell equations (Bonnor, 1961). This is particularly evident for axially symmetric fields, and one finds that the sets of equations governing the two cases can be transformed into the other by simple complex transformations of the dependent variables (Perjes, 1968; Ward, 1974). Bonnor used this correspondence to derive a solution describing a static massive source carrying a magnetic dipole, from the Kerr solution (Bonnor, 1966).

In the realm of static axisymmetric solutions of the Einstein-Maxwell equations the Bonnor solution stands out because it is not a member of the Weyl electromagnetic class. Because of this special nature of the Bonnor solution, a much deeper analysis than has hitherto appeared in the literature is long overdue. One of the more interesting aspects of this work is the fact that there appears to be a close formal connection between the Bonnor solution and the field due to two magnetic monopoles of opposite sign, each carrying a positive mass numerically equal to the pole strength, symmetrically situated on the symmetry axis. (Geometrically speaking this would be the 'sum' of two Riessner-Nordstrom $e^2 = m^2$ particles.)

The paper is divided into three sections. Section 1 introduces the Bonnor solution and also contains a brief sketch of the properties of the solution. Section 2 makes use of the algebraic invariants of the Einstein-Maxwell field to try and differentiate between true singularities of the field and those due to an unfortunate choice of coordinates. Section 3 looks at the horizons of the Bonnor solution and some questions concerning goedesic completeness are discussed. In this section specialisation to the symmetry axis of the solution brings out the close connection between this solution and the Riessner-Nordstrom solutions mentioned earlier.

Section 1

The metric we are interested in is:

$$ds = -\frac{Y^2P^2}{Q^3Z}(dR^2 + Z\,d\theta^2) - Z\left(\frac{Y}{P}\right)^2 \sin^2\theta\, d\phi^2 + \left(\frac{P}{Y}\right)^2 dt^2 \tag{1.1}$$

where

$$\begin{aligned} P &\equiv R^2 - 2mR - b^2\cos^2\theta \\ Q &\equiv (R-m)^2 - (b^2+m^2)\cos^2\theta \\ Y &\equiv R^2 - b^2\cos^2\theta \\ Z &\equiv R^2 - 2mR - b^2 \end{aligned} \tag{1.2}$$

and in these coordinates $(R, \theta, \phi, t) \leftrightarrow (x^1, x^2, x^3, x^0)$ the electromagnetic field is obtained from $F_{\alpha\beta} = \kappa_{\alpha;\beta} - \kappa_{\beta;\alpha}$ through

$$\kappa_\alpha = (0, 0, 2mbRP^{-1}\sin^2\theta, 0) \tag{1.3}$$

where, the semicolon denotes covariant differentiation.

As $R \to \infty$ the metric tends to the Minkowski metric and the R^{-1} term in g_{00} shows that the mass of the source is $2m$. Also, at large distances from the source, the electromagnetic potential describes a magnetic dipole of strength $2mb$. In the latter part of this work some attempt is made to try and clarify how the source is formed from simpler sources. By consideration of the symmetry axis alone, we are led to conjecture that the field is produced by two magnetic poles of strengths $-m$ and m both with mass m situated on the symmetry axis separated by a coordinate distance $2b$. While this interpretation accounts for the field an observer would experience on the axis and in asymptotic regions it is not clear that this accounts for the field in other regions of space-time.

The determinant of the metric is

$$g = -\frac{Y^4P^4\sin^2\theta}{Q^3} \tag{1.4}$$

The metric displays singular behaviour at $Q = 0, P = 0, Z = 0, Y = 0$. By making the transformation

$$\rho = Z^{1/2}\sin\theta \qquad z = (R-m)\cos\theta \tag{1.5}$$

the metric (1.1) goes into

$$ds^2 = -\left(\frac{Y}{P}\right)^2 \left\{\left(\frac{P}{Q}\right)^4 (d\rho^2 + dz^2) + \rho^2 \, d\phi^2\right\} + \left(\frac{P}{Y}\right)^2 dt^2 \qquad (1.6)$$

Using this form, a rather long calculation (Ward, 1974) shows that the solution has Weyl tensor which is type [1111] in the Penrose notation.

For $m = 0$ the metric (1.1) becomes

$$ds^2 = -\frac{(R^2 - b^2 \cos^2 \theta)}{R^2 - b^2}(dR^2 + [R^2 - b^2] \, d\theta^2) - (R^2 - b^2) \sin^2 \theta \, d\phi^2 + dt^2 \qquad (1.1a)$$

which is flat space-time in prolate spheroidal coordinates. That it is flat space-time is most easily seen from (1.6). This is to be compared with the Kerr solution: there, putting $m_{\text{Kerr}} = 0$ gave flat space-time in oblate spheroidal coordinates.

It is useful to represent the hypersurfaces of interest $Q = 0, P = 0, Z = 0, Y = 0$ (all at $t =$ const.) using these flat prolate coordinates. We note first:

$$\begin{aligned} Z &= 0 \to R = (R_1)_\pm = m \pm (m^2 + b^2)^{1/2} \\ P &= 0 \to R = (R_2)_\pm = m \pm (m^2 + b^2 \cos^2 \theta)^{1/2} \\ Q &= 0 \to R = (R_3)_\pm = m \pm |(m^2 + b^2)^{1/2} \cos \theta| \\ Y &= 0 \to R = (R_4)_\pm = \pm |b \cos \theta| \end{aligned} \qquad (1.7)$$

and for $m > 0$

$$(R_1)_+ \geqslant (R_2)_+ \geqslant (R_3)_+ > (R_4)_+ \qquad (1.8)$$

the equality signs occurring when $\cos \theta = 1$. There are two cases of interest:

Case 1. $b > 2m > 0$

We represent surfaces of interest in flat polar coordinates (ρ, z) given by taking $m = 0$ in (1.5), i.e.

$$\rho = (R^2 - b^2)^{1/2} \sin \theta \qquad z = R \cos \theta$$

then

(a) $$Y = 0 \to R = \pm |b \cos \theta|$$

Only $R = b$ $\theta = 0, \pi$ are represented in these coordinates. We shall see later that $Y = 0$ is singular. Thus $Y = 0$ gives two singular points on the ends of the 'rod' $R = b$. Again this is to be compared with the Kerr solution. There the surface corresponding to $Y = 0$ is also singular and is represented in flat space oblate spheroidal coordinates by a ring. (The ring of course bears the same relation to oblate coordinates as do the two points at the end of the rod to prolate coordinates.)

(b) $$Z = 0 \to R = m \pm (m^2 + b^2)^{1/2}$$

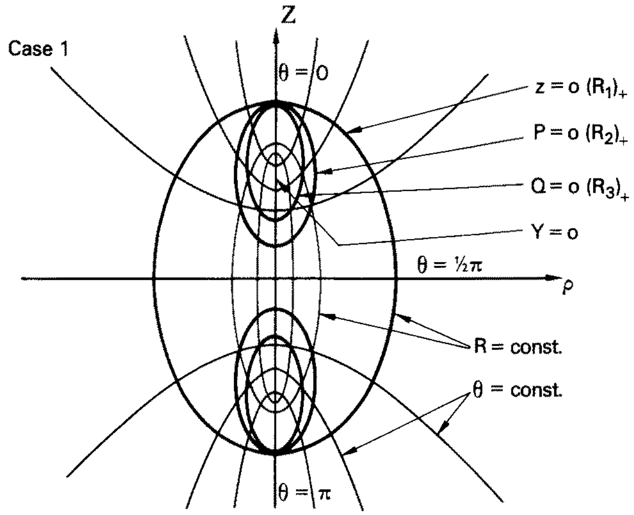
Case 1
Z
θ = 0
z = o (R1)+
P = o (R2)+
Q = o (R3)+
Y = o
θ = ½π
ρ
R = const.
θ = const.
θ = π

Fig. 1

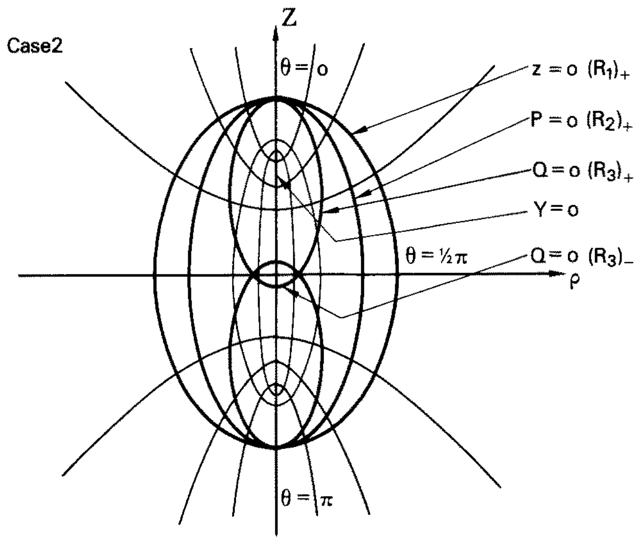
Case2
Z
θ = o
z = o (R1)+
P = o (R2)+
Q = o (R3)+
Y = o
θ = ½π
Q = o (R3)–
ρ
θ = π

Fig. 2

Only $R = (R_1)_+$ appears on this diagram.

(c) $$P = 0 \rightarrow R = m \pm (m^2 + b^2 \cos^2 \theta)^{1/2}$$

None of $R = (R_2)_-$ is represented and points for which $\cos\theta <$ $[1 - (2m/b)]^{1/2}$ on $R = (R_2)_+$ are missing as well. There are two sheets.

(d) $$Q = 0 \rightarrow R = m \pm |(m^2 + b^2)^{1/2} \cos\theta|$$

Here $R = (R_3)_-$ does not appear and points for which $\cos\theta <$ $[1 - (2mb/(m^2 + b^2))]^{1/2}$ are also missing. There are two sheets.

Case 2. $m > b > 0$

(a) $Y = 0$, (b) $Z = 0$ have the same form as Case 1.

(c) $P = 0$. $(R_2)_-$ is still missing. $(R_2)_+$ is now one closed curve.

(d) $Q = 0$. $(R_3)_-$ now makes an appearance and joins continuously onto the $(R_3)_+$ branch. The $(R_3)_+$ branch has a cusp at $\theta = \frac{1}{2}\pi$.

I shall not dwell on these diagrams too long. We shall see in a later paragraph that these diagrams do not give a faithful representation of the geometry. (The opposite effect occurs in the Kerr solution.) For example, the proper circumference of the $Z = 0$ surface (at const. t, R) is zero.

Section 2

Algebraic Invariants of the Bonnor Solution

We shall use these to try and differentiate between true singularities and those which are only due to a bad choice of coordinates.

To evaluate the invariants the tetrad components of the Riemann, Maxwell and permutation tensors are required. If e^A_α denotes our tetrad (capital Roman letters denote tetrad components) then the tetrad components of any tensor $T_{\alpha\beta\gamma}\ldots$ are $T_{ABC}\ldots$ where

$$T_{ABC}\ldots = T_{\alpha\beta\gamma}\ldots e^\alpha_A e^\beta_B e^\gamma_C$$

and since $e^\alpha_A e^B_\alpha = \delta^B_A$ then any contraction over tensor indices is equivalent to a contraction over tetrad indices.

We can arrive at the tetrad components of the Riemann tensor directly by using the method of differential forms (Israel, 1970). Thus choose 1-forms θ^A by

$$\theta^0 = \frac{P}{Y}\,dt \qquad \theta^1 = \frac{YP}{Q^{3/2}Z^{1/2}}\,dR \qquad \theta^2 = \frac{YP}{Q^{3/2}}\,d\theta \qquad \theta^3 = \frac{Z^{1/2}Y\sin\theta}{P}\,d\phi$$

These 1-forms define a tetrad by $\theta^A = e^A_\alpha\, dx$ giving

$$e^1_\alpha = \left(\frac{YP}{Q^{3/2}Z^{1/2}}, 0, 0, 0\right) \qquad e^2_\alpha = \left(0, \frac{YP}{Q^{3/2}}, 0, 0\right)$$

$$e_\alpha^3 = \left(0, 0, \frac{Z^{1/2} Y \sin\theta}{P}, 0\right) \qquad e_\alpha^0 = \left(0, 0, 0, \frac{P}{Y}\right)$$

Then using Cartan's equations (Israel, 1970) we easily find the tetrad components of the Riemann tensor are:

$$R^0{}_{110} = \frac{Q^{3/2} Z^{1/2}}{YP^2} A_{,1} + A^2 + BC$$

$$R^0{}_{120} = \frac{Q^{3/2}}{YP} A_{,2} + AB + DB$$

$$R^0{}_{220} = \frac{Q^{3/2}}{YP} B_{,2} + B^2 - AD$$

$$R^0{}_{330} = AE + BF$$

$$R^1{}_{221} = \frac{Q^{3/2}}{YP} C_{,2} - \frac{Q^{3/2} Z^{1/2}}{YP} D_{,1} + C^2 + D^2$$

$$R^1{}_{313} = -\frac{Q^{3/2} Z^{1/2}}{YP} E_{,1} - E^2 - CF$$

$$R^1{}_{323} = -\frac{Q^{3/2}}{YP} E_{,2} - EF - DF$$

$$R^2{}_{323} = -\frac{Q^{3/2}}{YP} F_{,2} - F^2 + DE$$

also from the formula $F^{AB} = e_\alpha^A e_\beta^B F^{\alpha\beta}$ we find

$$F^{31} = -\frac{Q^{3/2} \sin\theta}{Y^2 P^2} (R^2 + b^2 \cos^2\theta) mb$$

$$F^{32} = 4Rmb \cos\theta \frac{Z^{1/2} Q^{3/2}}{Y^2 P^2} \tag{2.1}$$

where

$$A = \frac{Q^{3/2} Z^{1/2}}{P^2 Y} P_{,1} - \frac{Q^{3/2} Z^{1/2}}{PY^2} Y_{,1}$$

$$B = \frac{Q^{3/2}}{P^2 Y} P_{,2} - \frac{Q^{3/2}}{PY^2} Y_{,2}$$

$$C = \frac{Q^{3/2}}{Y^2 P} Y_{,2} + \frac{Q^{3/2}}{YP^2} P_{,2} - \frac{3Q^{1/2}}{2PY} Q_{,2}$$

$$D = -\frac{Q^{3/2}Z^{1/2}}{Y^2P} Y_{,1} - \frac{Q^{3/2}Z^{1/2}}{YP^2} P_{,1} + \frac{3Q^{1/2}Z^{1/2}}{2YP} Q_{,1}$$

$$E = \frac{Q^{3/2}}{2YPZ^{1/2}} Z_{,1} + \frac{Z^{1/2}Q^{3/2}}{Y^2P} Y_{,1} - \frac{Z^{1/2}Q^{3/2}}{P^2Y} P_{,1}$$

$$F = \frac{Q^{3/2}}{PY} \cot\theta + \frac{Q^{3/2}}{Y^2P} Y_{,2} - \frac{Q^{3/2}}{P^2Y} P_{,2} \qquad (2.2)$$

Now it is easy to show (Israel, 1970) for the particular tetrad we have chosen that the tetrad components of the permutation tensor $\eta_{\alpha\beta\gamma\delta}$ are

$$\eta_{ABCD} = \pm\, \epsilon_{ABCD}$$

where ϵ_{ABCD} is the four-dimensional Levi-Civita symbol.

The algebraic invariants of an Einstein-Maxwell field have been given in spinor form by Penrose (1960). It is a straightforward matter to put these in tensor form. However, for our purposes the explicit form of these invariants is not important. We need only note that the invariants only involve products of the Riemann, Maxwell and permutation tensors, and so in tetrad form only involve products of the tetrad components of the Riemann and Maxwell tensors. Also, since for our particular tetrad, tetrad indices are raised and lowered with the Minkowski metric, it follows that if the tetrad components of the Riemann and Maxwell tensors are well behaved then so are *all* the invariants.

The invariant K (here $K = F_{\alpha\beta}F^{\alpha\beta}$)

$$K = F^{31}F_{31} + F^{32}F_{32}$$

$$= \frac{4Q^3}{Y^4P^4} [\sin^2\theta(R^2 + b^2\cos^2\theta) + 4\cos^2\theta R^2 Z]$$

Thus the space-time is singular at $Y = 0$. Space-time is also singular at $P = 0$ $\sin\theta \neq 0$.

I will now show that the invariants are well behaved on the $Z = 0$ and $Q = 0$ ($Q \neq Y$) surfaces. For this, all we have to show is that the tetrad components of the Maxwell and Riemann tersors are well behaved.

Noting (1.8) it is straightforward to show that the following behaviour of $A - F$ holds as $Z \to 0$ ($\sin\theta \neq 0$)

$$A_{,1} = \tfrac{1}{2}Z^{-1/2}\left\{\frac{Q^{3/2}}{P^2Y} P_{,1} - \frac{Q^{3/2}}{PY^2} Y_{,1}\right\} Z_{,1} + O(Z^{1/2}) \qquad A_{,2} = O(Z^{1/2})$$

$$B_{,1} = O(1) \qquad B_{,2} = O(1) \qquad C_{,1} = O(1) \qquad C_{,2} = O(1)$$

$$D_{,1} = -\tfrac{1}{2}Z^{-1/2}\left\{\frac{Q^{3/2}}{Y^2P} Y_{,1} + \frac{Q^{3/2}}{YP^2} P_{,1} - \frac{3Q^{1/2}}{2YP} Q_{,1}\right\} Z_{,1} + O(Z^{1/2})$$

$$D_{,2} = O(Z^{1/2}) \qquad E_{,1} = -\frac{1}{4}\frac{Z^{-3/2}Q^{3/2}}{PY}(Z_{,1})^2 + O(Z^{-1/2})$$

$$E_{,2} = \tfrac{1}{2}Z^{-1/2}\left\{\frac{3Q^{1/2}}{2YP}Q_{,2} - \frac{Q^{3/2}}{Y^2P}Y_{,2} - \frac{Q^{3/2}}{YP^2}P_{,2}\right\}Z_{,1} + O(Z^{1/2})$$

$$F_{,1} = O(1) \qquad F_{,2} = O(1) \tag{2.3}$$

These expansions imply the following behaviour on the tetrad components of the Riemann tensor:

$$R^0{}_{110} = O(1) \qquad R^0{}_{120} = O(Z^{1/2}) \qquad R^0{}_{220} = O(1) \qquad R^2{}_{323} = O(1)$$

$$R^0{}_{330} = O(1) \qquad R^1{}_{221} = O(1) \qquad R^1{}_{313} = O(1)$$

$$R^1{}_{323} = -\frac{1}{2}\frac{Z^{-1/2}Q^{3/2}}{PY}\left\{\frac{3Q^{1/2}}{2YP}Q_{,2} + \frac{Q^{3/2}}{PY}\cot\theta - \frac{2Q^{3/2}}{P^2Y}P_{,2}\right\}Z_{,1} + O(1) \tag{2.4}$$

So the only component which gives trouble is the one with three different indices $R^1{}_{323}$. However, substituting for P, Y and Q and noting that

$$P = Z + b^2\sin^2\theta \qquad Q = Z + (m^2 + b^2)\sin^2\theta$$

it is easily seen that

$$\frac{3Q^{1/2}}{2YP}Q_{,2} + \frac{Q^{3/2}}{YP}\cot\theta - \frac{2Q^{3/2}}{P^2Y}P_{,2} = \frac{Q^{1/2}\cot\theta}{P^2Y}\,[Z^2 + Z\sin^2\theta(4m^2 + b^2)]$$

so $R^1{}_{323} = O(1)$. Thus all the tetrad components of Riemann and Maxwell tensors are well behaved on the surface $Z = 0$ ($\sin\theta \neq 0$). It is obvious from (2.1) that the tetrad components are well behaved on the surface $Q = 0$ ($Q \neq Y$, $\sin\theta \neq 0$).

A more detailed calculation involving (2.4) shows that nothing untoward happens to the tetrad components of the Riemann and Maxwell tensors as the axis $\sin\theta = 0$ is approached on the surface $Z = 0$. Other approaches to $Z = 0$ $\sin\theta = 0$ seem far too complicated to look at in any detail.

Thus a study of the invariants have provided a fair deal of information. We have that the space-time is singular at $Y = 0$; $P = 0$ $\sin\theta \neq 0$ and no unreasonable behaviour (from the point of view of the invariants) occurs on the surfaces $Z = 0$; $Q = 0$ $Q \neq Y$. Of course, for the last two cases we cannot conclude that the space-time reflects the regularity displayed in the invariants. (There are many counter examples, cf. Sackfield (1972) and Ward (1974).)

Section 3. *Horizons*

The Bonnor solution admits two obvious Killing vectors

$$\underset{\phi}{\xi}^{\alpha} = (0, 0, 1, 0) \qquad \underset{t}{\xi}^{\alpha} = (0, 0, 0, 1) \tag{3.1}$$

Define $\omega_{\alpha\beta} \equiv \underset{\phi}{\xi}_{[\alpha}\underset{t}{\xi}_{\beta]}$ then Hawking & Ellis (1973) locally characterise the horizon† as those points for which $\omega^{\alpha\beta}\omega_{\alpha\beta} = 0$ (off the axis $\sin\theta = 0$). For stationary metrics admitting Killing vectors (3.1)

$$\omega^{\alpha\beta}\omega_{\alpha\beta} = g_{00}g_{33} - g_{30}^2 = Z\sin^2\theta \qquad \text{for Bonnor} \tag{3.2}$$

So off the axis $Z = 0$ represents the horizon. This surface, is of course, null $g^{\alpha\beta}Z_{,\alpha}Z_{,\beta} = 0$ on $Z = 0$. (As a slight digression we note that if Weyl-type coordinates are used, cf. (1.5), then this essentially means that one chooses radial coordinate $\rho^2 = g_{00}g_{33} - g_{30}^2$ and so the horizon will seem to be part of the axis itself. Thus any parts of space-time within the Killing horizon will not appear in Weyl coordinates. Cf. the Schwarzschild solution where the horizon $r = 2m$ in the usual coordinates appears as a rod on the axis in Weyl coordinates.)

From the point of view of the invariants $Z = 0$ seems to be non-singular. Is it compact? An indication that it is follows from a computation of the proper surface area of the surface.

The proper surface area of R = const., t = const. surfaces is given by the formula:

$$A = \int_0^{\pi}\int_0^{2\pi} \frac{Z^{1/2}Y^2}{Q^{3/2}}\sin\theta\, d\theta\, d\phi$$

$$= -4\pi Z^{1/2}\int_{\frac{1}{2}\pi}^{0} \frac{Y^2}{Q^{3/2}}\sin\theta\, d\theta$$

$$= 4\pi Z^{1/2}\int_0^1 \frac{(R^2 - b^2x^2)^2\, dx}{[(R-m)^2 - (M^2 + b^2)x^2]^{3/2}}$$

and after a long calculation we find

$$A = 4\pi\left\{\frac{R^4}{(R-m)^2} - \frac{b^4}{2(m^2+b^2)} - \frac{1}{(m^2+b^2)}\left[2R^2b^2 - \frac{3(R-m)^2b^4}{2(m^2+b^2)}\right]\right\}$$

$$-4\pi\frac{Z^{1/2}}{(m^2+b^2)^{3/2}}\left\{2R^2b - \frac{3(R-m)^2b^4}{2(m^2+b^2)}\right\}\left\{\sin^{-1}\left(-\frac{(m^2+b^2)^{1/2}}{R-m}\right)\right\} \tag{3.3}$$

† This, for stationary axisymmetric space-times which are regular predictable (no naked singularities) in which $\omega_{[\alpha\beta;\gamma}\omega_{\rho]\sigma} = 0$. This last condition is true for vacuum Einstein–Maxwell fields.

So the proper surface area of $Z = 0$ is

$$A_{Z=0} = 16\pi m^2 \frac{(R_1)_+^2}{(m^2 + b^2)} \tag{3.4}$$

So apparently the $Z = 0$ surface is compact. However, it is obvious from (1.1) that the proper circumference of the $Z = 0$ surface is zero. This result with (3.4) indicates some sort of singular behaviour at the poles and consequently the $Z = 0$ surface is not a regular horizon. However, it is still not clear to me whether or not the Bonnor solution has naked singularities. This result also disproves the schematic view of the $Z = 0$ surface we had earlier and suggests that the Weyl coordinates given in (1.5) give a more realistic description of the geometry.

Geodesic Completeness

The geodesic equations show incompleteness at all the surfaces $Q = 0$, $Z = 0, P = 0$, and $Y = 0$. Because of the rod-like nature of the $Z = 0$ surface there does not seem much point in trying to extend these incomplete geodesics through the $Z = 0$ surface (and thus extendability through the $Q = 0$ surface loses much interest).

When restricted to the axis $\sin \theta = 0$, the metric (1.1) becomes

$$ds^2 = -\left\{\frac{R^2 - 2mR - b^2 \cos^2 \theta}{R^2 - b^2 \cos^2 \theta}\right\}^{-2} dR^2 + \left\{\frac{R^2 - 2mR - b^2 \cos^2 \theta}{R^2 - b^2 \cos^2}\right\}^2 dt^2 \tag{3.5}$$

where $\cos^2 \theta = 1$.

I leave the functions of θ in the metric to avoid confusion as to which part of the axis, either $\theta = 0$ or $\theta = \pi$, a given value of R refers. Again we only consider values of R for which $R > b$ (this since $R = b\ \theta = 0, \pi$ singular). That (3.5) is singular at $R = b\ \theta = 0, \pi$ can easily be seen by examining the intrinsic Gaussian curvature of the two-surface (Walker, 1970)

$$G = \frac{1}{2}\frac{d^2F}{dR^2} \qquad F = \left\{\frac{R^2 - 2mR - b^2 \cos^2 \theta}{R^2 - b^2 \cos^2 \theta}\right\}^2 \qquad \cos^2 \theta = 1$$

we find

$$G = \frac{4m}{(R - b^2 \cos^2 \theta)^4} (-R^5 + 3mR^4 - 2bR + 8mbR + 3b^4R + mb^4)$$

So the surface is singular at $R = b$, $\theta = 0, \pi$. To show that the surface is regular at $R^2 - 2mR - b^2 = 0$ introduce a null coordinate by

$$du = \frac{dR}{F} + dt$$

then (3.5) becomes

$$ds^2 = F\, du^2 - 2\, du\, dR \tag{3.6}$$

which is non-singular at $F = 0$. The geodesic equations for (3.6) are

$$\frac{du}{d\lambda} = \frac{1}{F}(E \pm (E^2 - F\epsilon)^{1/2})$$
$$\frac{dR}{d\lambda} = \pm (E^2 - F\epsilon)^{1/2} \tag{3.7}$$

where λ is an affine parameter and $\epsilon = 1, -1, 0$ represents time-like, space-like and null geodesics respectively. E is a constant representing the energy in the time-like case.

If $du/d\lambda$, $dR/d\lambda$ were bounded for all R then this two-dimensional space-time would be geodesically complete because then each geodesic may be continued to infinite values of the affine parameter. From (3.7) we see that $dR/d\lambda$ is bounded for all $R \neq b$ (incompleteness at $R = b$ need not worry us as the space-time is singular there). However, $du/d\lambda$ is unbounded at $R^2 - 2mR - b^2 = 0$ which is a regular part of the space-time. To see that the space-time can be analytically continued through these points one need only note that the null form of the metric (3.6) is, in its qualitative features, essentially the same as the null form of the symmetry axis of the $e^2 = m^2$ Riessner-Nordstrom solution given by Carter (1966). The Riessner-Nordstrom solution is

$$ds^2 = -\left\{\frac{R^2 - 2mR + e^2}{R^2}\right\}^{-1} dR^2 - R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$+\left\{\frac{R^2 - 2mR + e^2}{R^2}\right\} dt^2 \tag{3.8}$$

which, when restricted by $\sin\theta = 0$ and $e^2 = m^2$, becomes

$$ds^2 = -\left\{\frac{R - m}{R}\right\}^{-2} dR^2 + \left\{\frac{R - m}{R}\right\}^2 dt^2 \tag{3.9}$$

and defining a null coordinate by

$$du = \left\{\frac{R}{R - m}\right\}^2 dR + dt$$

(3.9) becomes

$$ds = \left\{\frac{R - m}{R}\right\}^2 du^2 - 2\, du\, dR \tag{3.10}$$

Thus the correspondence between (3.6) and (3.10) is clear. If we concentrate on one branch of the axis at a time then the coefficient of du^2 in (3.6) has a repeated root at $R = m + (m^2 + b^2)^{1/2}$ and a singularity at $R = b$. The same characteristics are displayed in (3.10). Because of this, the same construction as used by Carter to extend (3.10) through $R = m$ can be used to extend (3.5) through $R = m + (m^2 + b^2)^{1/2}$ on either branch of the axis $\theta = 0, \pi$. Thus the Penrose-Carter diagram given for the symmetry axis of the Riessner-Nordstrom $e^2 = m^2$ solution will suffice, with only minor modifications as to the position of the singularity and the horizon, to describe the global features of each branch of the symmetry axis of the Bonnor solution.

Other similarities with the $e^2 = m^2$ Riessner-Nordstrom solution follow from this. For example both singularities of (3.9) and (3.5) are repulsive in the sense that no time-like geodesic can hit them. This is well known for the Riessner-Nordstrom solution and is easy to show for the Bonnor solution. From (3.7) as $R \to b$, i.e. $F \to \infty$, then we must choose $\epsilon = -1, 0$ to ensure $dR/d\lambda$ is real. This implies that no time-like geodesic can hit the singularity. We also note that (3.5) can be written

$$ds^2 = -\left\{1 - \frac{m}{R - b\cos\theta} - \frac{m}{R + b\cos\theta}\right\}^{-2} dR^2$$

$$+\left\{1 - \frac{m}{R - b\cos\theta} - \frac{m}{R + b\cos\theta}\right\}^{2} dt^2 \qquad (3.5a)$$

with $\cos^2\theta = 1$.

Thus close to $R = b, \theta = 0$ or $\theta = \pi$ this metric is essentially the same as an $e^2 = m^2$ Riessner-Nordstrom solution close to its singularity at $R = 0$.

Because of all these similarities with the Riessner-Nordstrom solution one is led to conjecture that the Bonnor solution (as regards the symmetry axis at least) has as its sources two Riessner-Nordstrom $e^2 = m^2$ particles at the ends of the rod $R = b$ (cf. diagram in earlier part of this section). We must look at the electromagnetic field to determine both the sign and the nature of the parameter e. (We remember that the geometry is independent of the sign of the charge and also of its nature. This last remark means that the parameter e^2 in the Riessner-Nordstrom solution can equally well describe a magnetic pole.)

The electromagnetic field as measured by an observer on the axis is given by F^{32} in (2.1).

$$H_R = F^{32} = \frac{4R\cos\theta\, mb}{(R^2 - b^2\cos^2\theta)^2} = \frac{m}{(R - b\cos\theta)^2} - \frac{m}{(R + b\cos\theta)^2}$$

at $\cos^2\theta = 1$

that is, the field is produced (where F^{32} is singular) by two magnetic poles of strengths m and $-m$ at $R = b, \theta = 0, \pi$ respectively.

So if we restrict our considerations to the symmetry axis alone, it is fairly reasonable to suppose that the exterior field is produced by two magnetic poles of strengths m and $-m$ both carrying a mass m and separated by a coordinate distance $2b$ at the ends of the rod $R = b$. If this simple idea is accepted for the full space-time it is easily seen that it accounts for the field at large distances from the sources: that is the field will refer to a mass $2m$, no magnetic poles, and a magnetic dipole of strength $2mb$. There is still the problem of how the two poles are kept apart. Some sort of strut is required and it is not clear how this manifests itself in the full solution. This simple interpretation does not throw any light on the singular surface $P = 0$.

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References

Bonnor, W. B. (1961). *Zeitschrift für Physik*, **161**, 439.
Bonnor, W. B. (1966). *Zeitschrift für Physik*, **190**, 444.
Carter, B. (1966). *Physical Review Letters*, **21**, 423.
Hawking, S. W. and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time.* Cambridge University Press, Cambridge.
Israel, W. (1970). *Communications of the Dublin Institute of Advanced Studies*, Series A, No. 19.
Penrose, R. (1960). *Annals of Physics*, **10**, 171.
Perjes, Z. (1968). *Nuovo Cimento*, **55B**, 600.
Sackfield, A. (1972). *Proceedings of the Cambridge Philosophical Society*, **71**, 131.
Walker, M. (1970). *Journal of Mathematical Physics*, **11**, 2280.
Ward, J. P. (1974). Thesis submitted to the University of London for the degree of Ph.D.